Technische Universität Ilmenau
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Quadratic Forms on Graphs and Maximum Weighted Induced Subgraphs

Bachelorarbeit zur Erlangung des akademischen Grades Bachelor of Science

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Abstracts

Abstract

Let $G = (V, E)$ be a simple, finite, undirected graph with vertex set $V(G)$ and edge set $E(G)$. An independent set of G is a subset I of vertices with no two of its members adjacent in G; a well-studied combinatorial optimization problem is to find an independent set of maximal cardinality.

Suppose there are weights assigned to the vertices; then a further issue is to find an independent set having the largest total weight. This is called the *maximum weight independent set problem*. The decision version of these problems are known to be NP-complete, and therefore the discussion of bounds on the maximum total weight of independent sets is justified. A known lower bound this work builds on was published by Gibbons, Hearn, Pardalos and Ramana in [6].

This Bachelor Thesis investigates quadratic forms on graphs motivated by a result of Motzkin and Straus [13]. Using these results, an improvement of the bound by Gibbons et al. is derived. Moreover, a generalization of weighted independence towards induced subgraphs of maximum weight is presented and investigated.

Zusammenfassung

Gegeben sei ein einfacher, endlicher, ungerichteter Graph $G = (V, E)$ mit Eckenmenge $V(G)$ und Kantenmenge $E(G)$. Eine unabhängige Menge von G ist eine Teilmenge von $V(G)$, in der je zwei Ecken nicht adjazent in G sind. Ein oft untersuchtes kombinatorisches Optimierungsproblem ist die Frage nach einer unabhängigen Menge maximaler Kardinalität.

Angenommen, den Ecken sind Gewichte zugewiesen, dann ist ein weiteres Problem, eine unabhängige Menge maximalen Gewichtes zu finden. Es ist bekannt, dass die Entscheidungsversion dieser Probleme NP-vollständig ist, wodurch Untersuchungen von Schranken für das maximale Gewicht unabhängiger Mengen gerechtfertigt sind. Eine bekannte Schranke, auf dem diese Arbeit aufbaut, ist eine Veröffentlichung von Gibbons, Hearn, Pardalos und Ramana [6].

Diese Bachelorarbeit untersucht quadratische Formen auf Graphen, welche durch eine Arbeit von Motzkin und Straus [13] motiviert sind. Mit den Ergebnissen kann die Schranke von Gibbons und andere verbessert werden. Des Weiteren wird eine Verallgemeinerung der gewichteten Unabhängigkeit zu maximal gewichteten induzierten Untergraphen untersucht.

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Contents

1 Introduction and Results

The *independent set problem* is a well-known example of a combinatorial optimization problem. Given a *simple*, *finite*, *undirected graph* $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$, the problem is to find an independent set of maximal cardinality. An independent set is a subset I of vertices of G with no two of its members adjacent in G; the maximum cardinality is denoted by the *independence number* $\alpha(G)$.

The independence number belongs to the most fundamental and well-studied graph parameters; not only because the decision version of it was one of the first problems shown to be NP-complete in Karp's original paper on computational complexity [9]. In view of its computational complexity [4, 8] it is justified to deal with bounds on these numbers, primarily with lower bounds. Many known lower bounds for the independence number can be found in literature and there are internet projects for it [12].

The results of this Bachelor Thesis will be published as a joint work with Prof. Dr. Jochen Harant [7].

1.1 Results

As an approach to get bounds, various attempts have been made to replace the combinatorial optimization problems by continuous ones.

An example is the following result of T. S. Motzkin and E. G. Straus [13]:

Statement 1 (Result of T. S. Motzkin and E. G. Straus [13])

There is an independent set I of G such that

$$|I| \geq \frac{\left(\sum\limits_{i \in V(G)} x_i\right)^2}{\sum\limits_{i \in V(G)} x_i^2 + 2\sum\limits_{ij \in E(G)} x_i x_j} \tag{1.1}$$

for all real $x_i \geq 0$ with $i \in V(G)$, satisfying $\sum_i x_i \neq 0$.

A generalization of the maximum independent set problem arises when weights are assigned to the vertices. In this case the problem is to find an independent set of maximum weight. This is called the *maximum weight independent set problem*. Thus, let w_i be the weight of vertex $i \in V(G)$, and we denote the weight for a subset S of vertices of G as $w(S) = \sum\limits_{i \in S} w_i$.

Gibbons et al. generalised Statement 1 to weighted independence in their paper "Continuous characterizations of the maximum clique problem" [6]. Their result is the following:

Statement 2 (Result of L. E. Gibbons et al. [6])

Let $w_i > 0$ for $i \in V(G)$. Then there is an independent set I of G such that

$$w(I) \geq \frac{\left(\sum\limits_{i\in V(G)} x_i\right)^2}{\sum\limits_{i\in V(G)} \frac{x_i^2}{w_i} + \sum\limits_{ij\in E(G)} \left(\frac{1}{w_i} + \frac{1}{w_j}\right) x_i x_j} \tag{1.2}$$

for all real $x_i \geq 0$ with $i \in V(G)$, satisfying $\sum_i x_i \neq 0$.

In the following we investigate the impact of the result in Statement 2 with regard to some improvements or generalizations. Two main results will be obtained.

It is possible to improve Statement 2; this is denoted in Theorem 1. Another question arising is whether there is a concept of generalizing the weighted independence. A possible approach is shown later, and a bound comparable to Statement 2 is provided in Theorem 2.

Theorem 1

Let $w_i > 0$ for $i \in V(G)$. Then there is an independent set I of G such that

$$w(I) \geq \frac{\left(\sum\limits_{i\in V(G)} x_i\right)^2}{\sum\limits_{i\in V(G)} \frac{x_i^2}{w_i} + \sum\limits_{ij\in E(G)} \max\left\{\frac{1}{2w_i} + \frac{1}{w_j}, \frac{1}{w_i} + \frac{1}{2w_j}, \min\left\{\frac{2}{w_i}, \frac{2}{w_j}\right\}\right\} x_i x_j} \tag{1.3}$$

for all real $x_i \geq 0$ with $i \in V(G)$, satisfying $\sum_i x_i \neq 0$.

Note that $\max\left\{\frac{1}{2w_i} + \frac{1}{w_j}, \frac{1}{w_i} + \frac{1}{2w_j}, \min\left\{\frac{2}{w_i}, \frac{2}{w_j}\right\}\right\} \leq \frac{1}{w_i} + \frac{1}{w_j}$ for all $ij \in E(G)$; hence, the denominator is smaller in case that there are edges $ij \in E(G)$ with $w_i \neq w_j$, and (1.3) would be an improvement of (1.2) in that case.

For the convenience of the reader, we skip the explanation of the used terminology for the rest of this section at this point and go on by facing the question of generalised weighted independence. The reader is referred to section 1.2 for notations and terminology.

Obviously, a vertex set I of G is independent if and only if the subgraph $G[I]$ induced by I is K_2-free. An approach would be to characterise the "generalised independent sets" by $\mathcal{F}$-free induced subgraphs. Instead of edgeless subgraphs, we will investigate the problem of finding an $\mathcal{F}$-free induced subgraph H of G of maximum weight $w(H)$.

That this idea is favourable can be argued because there are many classes of graphs that can be described in terms of a collection of forbidden subgraphs. For further details, we refer the reader to collections of graph classes [2, 14].

Note that the complementary graph $\overline{G}$ of G contains an induced subgraph $\overline{H}$ of weight $w(\overline{H}) = w(H)$ if G contains an induced subgraph H of weight $w(H)$, i. e. the translation of the following results into the "complementary version" will be omitted.

Let the real numbers b_{ij} be the weight of the edge $ij \in E(G)$. For F being an arbitrary graph we define the condition $C(F)$ linking a restriction on the parameters w_i and b_{ij} to a property of the induced subgraph H.

Definition ($C(F)$)

For each induced subgraph U of G isomorphic to F there is a proper vertex partition of $V(U)$ into V_1 and V_2 fulfilling the inequality

$$\nu_2^2 \sum_{i\in V(U_1)} \frac{1}{w_i} + \nu_1^2 \sum_{i\in V(U_2)} \frac{1}{w_i} + \nu_2^2 \sum_{ij\in E(U_1)} b_{ij} + \nu_1^2 \sum_{ij\in E(U_2)} b_{ij} \leq \nu_1\nu_2 \sum_{ij\in E^*} b_{ij}, \tag{$C(F)$}$$

where $U_1 = U[V_1]$ and $U_2 = U[V_2]$ denote the induced subgraphs on $\nu_1 \geq 1$ and $\nu_2 \geq 1$ vertices, respectively. $E^* = \{ij \in E(U) \ : \ i \in V_1, j \in V_2\}$ is the set of edges of U between V_1 and V_2.

Even if this condition seems to be complicated at first sight, it is easily manageable as one can see in section 1.3, when the impact of it will be exemplified alongside some simplifications. Note that if G itself is F-free, then there is no induced subgraph U in G isomorphic to F; thus, $C(F)$ is fulfilled.

For the generalised weighted independence problem a lower bound on the weight of a maximum weighted $\mathcal{F}$-free induced subgraph H is denoted in the following Theorem 2.

Theorem 2

Let $\mathcal{F} = \{F_1, \dots, F_p\}$ be a finite family of graphs with $K_1 \notin \mathcal{F}$, and let $w_i > 0$ for $i \in V(G)$ and $b_{ij} \geq 0$ for $ij \in E(G)$ be the weights on the vertices and egdes. Denote the minima over the edge and vertex weights by $b = \min_{ij\in E(G)} b_{ij}$ and $w = \min_{i\in V(G)} w_i$, respectively.

If $C(F_1), \dots, C(F_p)$ are fulfilled, then there is an $\mathcal{F}$-free induced subgraph H of G such that

$$w(H) - \frac{bw}{2+bw} \cdot \sum_{\substack{i\in V(H),\\ d_H(i)\geq 1}} w_i \geq \frac{\left(\sum\limits_{i\in V(G)} x_i\right)^2}{\sum\limits_{i\in V(G)} \frac{x_i^2}{w_i} + \sum\limits_{ij\in E(G)} b_{ij}x_ix_j} \tag{1.4}$$

for all real $x_i \geq 0$ with $i \in V(G)$, satisfying $\sum_i x_i \neq 0$.

We will clarify the proposition of this theorem after a short overview of used terminology. First, the condition $C(F)$ is investigated in section 1.3, some examples exemplify this condition and the meaning of F-free for certain graphs F is shown. In section 1.4, some known lower bounds are presented and compared with ones derived from Theorem 2 by suitably setting x_i for $i \in V(G)$. The proofs will be given in chapter 2.

1.2 Terminology

We use standard notation and terminology of graph theory; for further details, we refer the reader to "Combinatorial Optimization" by Korte and Vygen [11]. In the following some standard definitions are repeated.

Consider a *simple*, *finite*, *undirected graph* $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. The *neighbourhood* and the *degree* of $i \in V(G)$ are denoted by $N_G(i)$ and $d_G(i) = |N_G(i)|$, respectively. Let $\delta(G)$ and $\Delta(G)$ be the *minimum degree* and the *maximum degree* of G, respectively.

The *complementary graph* $\overline{G}$ of the graph G has vertex set $V(\overline{G}) = V(G)$ and edge set $ij \in E(\overline{G})$ if and only if $ij \notin E(G)$.

If $ij \in E$ for every pair ij with $i, j \in V$, then G is called a *complete graph*. A complete graph on $r \in \mathbb{N}$ vertices is denoted by K_r.

A *vertex partition* divides $V(G)$ into two sets V_1 and V_2 such that $V_1 \cup V_2 = V(G)$ and $V_1 \cap V_2 = \emptyset$. We say it is *proper* if $|V_1| \geq 1$ and $|V_2| \geq 1$.

A *bipartite graph* on A and B has a proper vertex partition into A and B with all edges $e \in E$ join vertices belonging to the distinct sets, i.e. $e = ij$ then $i \in A$, $j \in B$ or vice versa. A bipartite graph will be called to be *complete* if there is an edge $ij \in E$ for all $i \in A$, $j \in B$. It is denoted by $K_{a,b}$, where $|A| = a$ and $|B| = b$.

Let S be a subset of $V(G)$. A *subgraph* of G is a graph H with $V(H) = S \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is called an *induced subgraph* of G if $E(H) = \{xy \in E(G) \ : \ x, y \in S\}$, and it is denoted as $H = G[S]$.

Two graphs G and G' are called to be *isomorphic* if there are bijections $\phi_V : V(G) \to V(G')$ and $\phi_E : E(G) \to E(G')$ such that $\phi_E(vw) = \{\phi_G(v), \phi_G(w)\}$ for each edge $vw \in E(G)$.

Let $\mathcal{F}$ be a family of graphs. An induced subgraph of G is $\mathcal{F}$-free if it contains no induced subgraph being isomorphic to a member of $\mathcal{F}$. Since a graph with more vertices than G cannot be an induced subgraph of G, we assume that $\mathcal{F}$ is a finite set of finite, simple, undirected and pairwise non-isomorphic graphs. We write F-free instead of $\mathcal{F}$-free if $\mathcal{F} = \{F\}$.

For $i, j \in V$ an *i-j-walk* W in G is a sequence $(i = x_0, e_1, x_1, e_2, \ldots, x_{r-1}, e_r, x_r = j)$ with $r \geq 1$ such that $x_k \in V(G)$ for $k = 0, \ldots, s$, $e_k = x_{k-1}x_k \in E(G)$ for $k = 1, \ldots, r$ and $e_k \neq e_l$ for $k \neq l$. An *i-j-path* P is a subgraph $P = (\{x_0, \ldots x_r\}, \{e_1, \ldots, e_r\})$ of G if $(x_0, e_1, x_1, e_2, \ldots, x_{r-1}, e_r, x_r)$ is an i-j-walk with mutually distinct vertices x_k. The *length* of P is the number of vertices in P, that is $|V(P)|$. An i-j-path P will be called to be *shortest* if there is no i-j-path with a smaller length than P. An arbitrary path on $r \in \mathbb{N}$ vertices is denoted by P_r. A *circuit* C is a subgraph $C = (\{x_1, \ldots x_r\}, \{e_1, \ldots, e_r\})$ of G if $(x_0, e_1, x_1, e_2, \ldots, x_{r-1}, e_r, x_r)$ with $x_0 = x_r$ is a closed walk with mutually distinct vertices x_k. An arbitrary circuit on $r \in \mathbb{N}$ vertices is denoted by C_r.

A graph G is *connected* if there is an i-j-path for all $i, j \in V$. The maximal connected subgraph of G with respect to the number of vertices are its *components*.

A subset I of vertices of G is *independent* if no two of its members are adjacent in G. The induced subgraph $G[I]$ of G by I is called an *edgeless graph*. A *maximum independent set* is an independent set with most elements among all independent sets, its cardinality is denoted by the *independence number* $\alpha(G)$.

A subset J of vertices of G is called a *clique* of G if its induced subgraph $G[J]$ is a complete graph. The *clique number* $\omega(G)$ is the maximum cardinality among all cliques of G, which is the independence number of the complementary graph $\alpha(\overline{G})$.

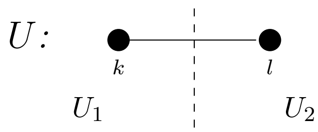
For a graph G let $w : V(G) \to \mathbb{R}$ be a *vector of vertex weights*. We shortly write $w_i := w(i)$ for $i \in V(G)$. Given a subset S of vertices of G and an induced subgraph H of G, we define the *weight of S and H* as $w(S) = \sum\limits_{i \in S} w_i$ and $w(H) = \sum\limits_{i \in V(H)} w_i$, respectively.

1.3 Simple edge conditions

In Theorem 2 the condition $C(F)$ is used. To make the reader familiar with the condition, let us consider a few examples.

Example 1.3.1 *We start with a small graph.*

For this, let F be a single edge, that is the K_2. There is only one way of dividing the vertices into two sets in order to get a proper partition.
Then $C(K_2)$ is fulfilled if for all $kl \in E(G)$



$$\frac{1}{w_k} + \frac{1}{w_l} \le b_{kl}$$

holds.

Remark 1.3.2 *A vertex set I of G is independent if and only if its induced subgraph $G[I]$ is edgeless, and this is the case if $G[I]$ is K_2-free. Hence, let $b_{ij} = \frac{1}{w_i} + \frac{1}{w_j}$ for $ij \in E(G)$, and Statement 2 follows directly from Theorem 2. Note that in this case the sum of the left side of the inequality is zero, and only $w(H)$ remains.*

Setting additionally $w_i = 1$ for all $i \in V(G)$, Statement 1 follows immediately.

The condition $C(F)$ is used for Theorem 2. If it is fulfilled, then there is an F-free induced graph H of G. Let us consider the condition $C(P_3)$ of a path on three vertices and its meaning for the subgraph H in Theorem 2.

Lemma 1.3.3 (Lemma about P_3-free graphs)

Let T be an arbitrary graph. If T does not contain P_3 as an induced subgraph, then T is the induced union of complete graphs, i. e. all of its components are complete graphs.

PROOF: Let T' be an arbitrary component of T. Assume T' is not complete, then there is $i, j \in V(T')$ with $ij \notin E(T')$. Let P be a shortest i-j-path.

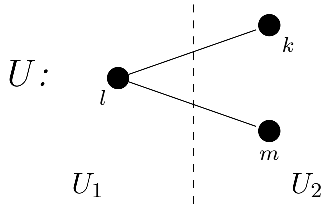
If P is of length 3, then P is an induced P_3 of T, a contradiction. Therefore the path $P = (i, e_1, x, e_2, y, e_3, \ldots, j)$ is at least of length 4. Consider $R = (i, e_1, x, e_2, y)$. If $iy \notin E(T)$, then T contains a P_3 as an induced subgraph, a contradiction. Else, $e' = iy \in E(T)$, and $P' = (i, e', y, e_3, \ldots, j)$ is a shorter i-j-path, a contradiction to the choice of P.

Thus, all components of T are complete graphs. ■

Example 1.3.4 *Let F be a path on three vertices, that is the P_3. If for every induced path $P = (\{k,l,m\},\{kl,lm\})$ of G on three vertices the inequality*

$$\frac{1}{w_k} + \frac{4}{w_l} + \frac{1}{w_m} \le 2(b_{kl} + b_{lm})$$

holds, then $C(P_3)$ is fulfilled, and H can be chosen as an induced union of complete subgraphs of G.



Remark 1.3.5 *Consider again $C(P_3)$:*

$$\frac{1}{w_k} + \frac{4}{w_l} + \frac{1}{w_m} \le 2(b_{kl} + b_{lm}).$$

It can be laborious to check whether $C(F)$ is fulfilled for given graphs G and F. Therefore, it may be useful to consider edge conditions. An edge condition *for an edge $kl \in E(G)$ is a simple condition on the three parameters a_k, a_l and b_{kl} such that if it holds for all edges in G then it implies $C(F)$. We try to replace all conditions $C(F)$ by these sufficient conditions because they directly propose a choice for the parameters b_{ij} in Theorem 2.*

For example if $b_{ij} \ge \max\left\{\frac{1}{2w_i} + \frac{1}{w_j}, \frac{1}{w_i} + \frac{1}{2w_j}\right\}$ holds for all $ij \in E(G)$, then $C(P_3)$ is fulfilled.

Using Theorem 2, the following lower bound on the weight of the union of cliques can be derived. Note that the b_{ij} are set according to the edge condition.

Remark 1.3.6 *Let $w_i > 0$ for $i \in V(G)$, and let w and b denote the minima of the vertex and edge weights, respectively; i. e.*

$$w = \min_{i \in V(G)} w_i, \qquad \text{and} \qquad b = \min_{ij \in E(G)} \max\left\{\frac{1}{2w_i} + \frac{1}{w_j}, \frac{1}{w_i} + \frac{1}{2w_j}\right\}.$$

Then there are $r \in \mathbb{N}$ and disjoint cliques $H_1, \ldots, H_r$ of G such that

$$\sum_{s=1}^{r} w(H_s) - \frac{bw}{2+bw} \cdot \sum_{\substack{s \in \{1,\ldots,r\},\\ |V(H_s)| \ge 2}} w(H_s) \ge \frac{\left(\sum\limits_{i \in V(G)} x_i\right)^2}{\sum\limits_{i \in V(G)} \frac{x_i^2}{w_i} + \sum\limits_{ij \in E(G)} \max\left\{\frac{1}{2w_i} + \frac{1}{w_j}, \frac{1}{w_i} + \frac{1}{2w_j}\right\} x_i x_j}$$

for all real $x_i \ge 0$ with $i \in V(G)$, satisfying $\sum_i x_i \neq 0$.

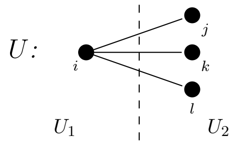
Proof: Set $b_{ij} = \max\left\{\frac{1}{w_i} + \frac{1}{2w_j}, \frac{1}{w_i} + \frac{1}{2w_j}\right\}$ for all $ij \in E(G)$, then $C(P_3)$ is fulfilled by Remark 1.3.5, and Theorem 2 can applied. ∎

Until now, we have only treated the condition $C(F)$ for one graph F. By combining several conditions it is possible to exclude a family $\mathcal{F}$ of subgraphs from being induced subgraphs of H.

For example consider some graphs on four vertices in Example 1.3.7. Let F be a complete K_4 that is missing an edge. If $C(K_{1,3})$, $C(C_4)$, $C(P_4)$ and $C(F)$ are fulfilled, then H can be chosen to be $\{K_{1,3}, C_4, P_4, F\}$-free. The subsequent Lemma 1.3.8 will show that all components of H are complete graphs or graphs consisting of two complete graphs with one common vertex.

Example 1.3.7 *If for every connected induced subgraph U of G on four vertices with vertex set $V(U) = \{i, j, k, l\}$ the subgraph U is isomorphic to …*

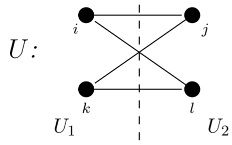
a claw $K_{1,3}$, *and*



$$\frac{9}{w_i} + \left(\frac{1}{w_j} + \frac{1}{w_k} + \frac{1}{w_l}\right) \leq 3(b_{ij} + b_{ik} + b_{il})$$

holds, where $d_U(i) = 3$, then $C(K_{1,3})$ is fulfilled.

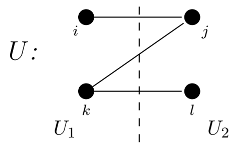
a circuit C_4, *and*



$$\frac{1}{w_i} + \frac{1}{w_j} + \frac{1}{w_k} + \frac{1}{w_l} \leq b_{ij} + b_{jk} + b_{kl} + b_{li}$$

holds, then $C(C_4)$ is fulfilled.

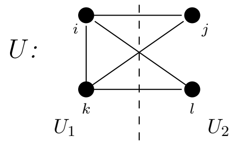
a path P_4, *and*



$$\frac{1}{w_i} + \frac{1}{w_j} + \frac{1}{w_k} + \frac{1}{w_l} \leq b_{ij} + b_{jk} + b_{kl}$$

holds, then $C(P_4)$ is fulfilled.

a complete subgraph that is missing an edge, *and*



$$\frac{1}{w_i} + \frac{1}{w_j} + \frac{1}{w_k} + \frac{1}{w_l} + b_{ik} \leq b_{ij} + b_{jk} + b_{kl} + b_{li}$$

holds, where $d_U(i) = d_U(k) = 3$, then $C(F)$ is fulfilled.

Lemma 1.3.8 (Lemma about some 4-vertex-graph-free subgraphs)

Let T be an arbitrary graph and F be a complete K_4 that is missing an edge. If T is $\{K_{1,3}, C_4, P_4, F\}$-free, then the components of T are complete graphs or graphs consisting of two complete graphs with one common vertex.

Proof: Let T' be an arbitrary component of T. Note that if $|T'| \leq 3$, T' is either K_1, K_2, P_3, or K_3. This would not be a contradiction to the assertion. Thus, let $|T'| \geq 4$.

If T' does not contain P_3 as induced subgraph, then, according to Lemma 1.3.3, T' is complete.

Else, consider a P_3 yxz.

Let Y be the neighbourhood of y except x, i. e. $Y = N_T(y) \setminus \{x\}$. Then Y is a subset of $N_T(x)$. Otherwise there is $a \in Y$, $a \notin N_T(x)$, thus $ayxz$ and $ayxza$ are a P_4 and a C_4, respectively, depending whether $az \in E(T)$ or not.

Further $Y \cup \{x, y\}$ is complete. Otherwise there is a missing edge $ij \notin E(Y)$, $i, j \neq x, y$. Then the induced subgraph $R = T'[\{x, y, i, j\}]$ is a K_4 without the missing edge ij, a contradiction. Analogously, $N_T(z) \cup \{z\}$ is complete, too.

Note that the neighbourhood of x is included in $N_T(y) \cup \{y\} \cup N_T(z) \cup \{z\}$, otherwise there would be another neighbour u of x, and $\{y, z, u\}$ are not adjacent. However, they would form a claw with x, a contradiction.

Consequently, the vertices of T' are $N_T(y) \cup \{y\}$ and $N_T(z) \cup \{z\}$. Consider the intersection $I := (N_T(y) \cup \{y\}) \cap (N_T(z) \cup \{z\})$, then $I = \{x\}$. Otherwise with $u \in I$, $uyxzu$ is a C_4, a contradiction.

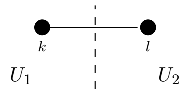
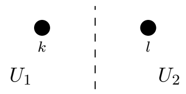
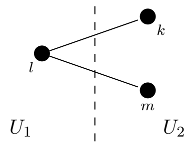
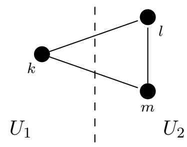
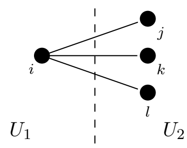
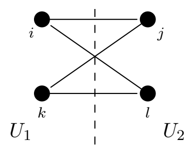
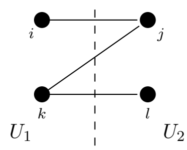
Thus, T' exists of two complete graphs with one common vertex x, and T is as asserted. ∎

Similarly to Remark 1.3.6, a bound can be describe for the weight of an induced subgraph whose components are complete graphs or graphs consisting of two complete graphs with one common vertex.

After considering $C(F)$ for some graphs, a summary of $C(F)$ for certain graphs F alongside some simple sufficient conditions is listed.

Table 1.3.9

This table lists for some small graphs the conditions $C(F)$ and a sufficient one, being an edge condition if possible.

F	condition $C(F)$	vertex partition	sufficient (simple edge) condition
K_2	$\frac{1}{w_k} + \frac{1}{w_l} \leq b_{kl}$	U: 	
$\overline{K_2}$	$\frac{1}{w_k} + \frac{1}{w_l} \leq 0$	U: 	$\frac{1}{w_i} < 0$ for all $i \in V(G)$
P_3	$\frac{1}{w_k}+\frac{4}{w_l}+\frac{1}{w_m} \leq 2(b_{kl}+b_{lm})$	U: 	$b_{ij} \geq \max\left\{\frac{1}{2w_i}+\frac{1}{w_j}, \frac{1}{w_i}+\frac{1}{2w_j}\right\}$ for all $ij \in E(G)$
K_3	$\frac{4}{w_k} + (\frac{1}{w_l}+\frac{1}{w_m}) + b_{lm} \leq$ $2(b_{kl}+b_{km})$	U: 	$2\left(\frac{1}{w_k}+\frac{1}{w_l}+\frac{1}{w_m}\right) \leq$ $b_{kl}+b_{km}+b_{lm}$ for all triangles $U = (\{k,l,m\},\{kl,km,lm\})$
$K_{1,3}$	$\frac{9}{w_i} + (\frac{1}{w_j}+\frac{1}{w_k}+\frac{1}{w_l}) \leq$ $3(b_{ij}+b_{ik}+b_{il})$	U: 	$b_{ij} \geq \max\left\{\frac{1}{3w_i}+\frac{1}{w_j}, \frac{1}{w_i}+\frac{1}{3w_j}\right\}$ for all $ij \in E(G)$
C_4	$\frac{1}{w_i}+\frac{1}{w_j}+\frac{1}{w_k}+\frac{1}{w_l} \leq$ $b_{ij}+b_{jk}+b_{kl}+b_{li}$	U: 	$b_{ij} \geq \frac{1}{2w_i}+\frac{1}{2w_j}$ for all $ij \in E(G)$
P_4	$\frac{1}{w_i}+\frac{1}{w_j}+\frac{1}{w_k}+\frac{1}{w_l} \leq$ $b_{ij}+b_{jk}+b_{kl}$	U: 	$b_{ij} \geq \max\left\{\frac{1}{w_i}, \frac{1}{w_j}\right\}$ for all $ij \in E(G)$

Finally, we will show that our approach also leads to reasonable results if G itself has a property characterised by forbidden induced subgraphs.

Remark 1.3.10 *Note that the condition $C(F)$ is also fulfilled if G itself is F-free.*

Similarly to Example 1.3.7, $b_{ij} \geq \frac{1}{2w_i} + \frac{1}{2w_j}$ for $ij \in E(G)$ implies that $C(F)$ holds for every even circuit F. Thus, if G is bipartite and $b_{ij} \geq \frac{1}{2w_i} + \frac{1}{2w_i}$ for $ij \in E(G)$ then H in Theorem 2 can be chosen to be a forest.

If additionally $b_{ij} \geq \max\{\frac{1}{3w_i} + \frac{1}{w_j}, \frac{1}{w_i} + \frac{1}{3w_j}\}$ holds for every $ij \in E(G)$ then, by Table 1.3.9, the subgraph H is $K_{1,3}$-free, that is claw-free. H is circuit-free and claw-free; hence, $d_H(i) \leq 2$ for $i \in V(H)$, and H is a linear forest.

Let the forest H consist of $r \in \mathbb{N}$ trees denoted by $H_1, \dots, H_r$, and let

$$c(U) = \begin{cases} 1, & |V(U)| = 1 \\ \frac{2}{2+bw}, & |V(U)| \geq 2 \end{cases}$$

for U being an induced subgraph of G.

Then we get the following bound from Theorem 2, where $b = \min\limits_{ij \in E(G)} b_{ij}$ and $w = \min\limits_{i \in V(G)} w_i$:

$$\sum_{s=1}^{r} c(H_s)w(H_s) \geq \frac{\left(\sum\limits_{i \in V(G)} x_i\right)^2}{\sum\limits_{i \in V(G)} \frac{x_i^2}{w_i} + \sum\limits_{ij \in E(G)} b_{ij}x_ix_j}$$

for all real $x_i \geq 0$ with $i \in V(G)$, satisfying $\sum_i x_i \neq 0$.

Remark 1.3.11 *As we have seen we can derive many classes of induced subgraphs by freely setting $\mathcal{F}$. But this is the only possibility to demand properties on the subgraphs. Thus, it is not possible to make some preconditions on the connectivity. The K_1 is always an $\mathcal{F}$-free subgraph; hence, we cannot get rid of the case $|V(H_s)| = 1$ in Remark 1.3.6 and 1.3.10.*

1.4 Known Bounds

In the proceeding section Theorem 2 is explained with regard to specify the searched subgraph, which is characterised by forbidden induced subgraphs; this is done by the conditions $C(F)$ leading to a choice for the edge weights b_{ij} of the edges $ij \in E(G)$.

In this section we show how to derive a lot of lower bounds on $w(H)$ by suitable choice of real numbers x_i for $i \in V(G)$.

There exists a great amount of bounds for the independence and clique number; they are proofed in different manners like probabilistic method as well as constructive by an algorithm. As an example, let us improve the famous bound

$$CW(G) = \sum_{i \in V(G)} \frac{1}{d_G(i) + 1}$$

in the case of graphs with at least one non-regular component. This bound was independently proofed by Y. Caro and V. Wei ([3, 17]).

Corollary 1.4.1

With $x_i = \frac{1}{d_G(i)+1}$ *for* $i \in V(G)$ *we obtain the following lower bound on* $\alpha(G)$ *from Statement 2:*

$$\alpha(G) \geq \frac{CW(G)^2}{CW(G) - \sum\limits_{ij \in E(G)} \left(\frac{1}{d_G(i)+1} - \frac{1}{d_G(j)+1} \right)^2}. \tag{1.5}$$

PROOF: Set $x_i = \frac{1}{d_G(i)+1}$ and $w_i = 1$ for $i \in V(G)$, then according to Statement 2:

$$\alpha(G) \geq \frac{\left(\sum\limits_{i \in V(G)} \frac{1}{d_G(i)+1} \right)^2}{\sum\limits_{i \in V(G)} \left(\frac{1}{d_G(i)+1} \right)^2 + 2 \sum\limits_{ij \in E(G)} \left(\frac{1}{d_G(i)+1} \right) \left(\frac{1}{d_G(j)+1} \right)}$$

$$= \frac{\left(\sum\limits_{i \in V(G)} \frac{1}{d_G(i)+1} \right)^2}{\sum\limits_{i \in V(G)} \left(\frac{1}{(d_G(i)+1)^2} + \frac{d_G(i)}{(d_G(i)+1)^2} \right) + 2 \sum\limits_{ij \in E(G)} \left(\left(\frac{1}{d_G(i)+1} \right) \left(\frac{1}{d_G(j)+1} \right) - \frac{1}{2} \left(\frac{1}{d_G(i)+1} \right)^2 - \frac{1}{2} \left(\frac{1}{d_G(j)+1} \right)^2 \right)}$$

$$= \frac{\left(\sum\limits_{i \in V(G)} \frac{1}{d_G(i)+1} \right)^2}{\sum\limits_{i \in V(G)} \frac{1}{d_G(i)+1} + 2 \sum\limits_{ij \in E(G)} \left(-\frac{1}{2} \left(\frac{1}{d_G(i)+1} - \frac{1}{d_G(j)+1} \right)^2 \right)},$$

where the first equality sign makes use of

$$\sum_{i \in V(G)} d_G(i) t_i = \sum_{ij \in E(G)} (t_i + t_j)$$

for any real numbers t_i with $i \in V(G)$. This equation can be considered as the simplest version of the charging-discharging method. ∎

Next, we will consider the maximum weight independent set problem. Some known results are provided and Statement 2 leads to some improvements of existing lower bounds. For this, let I be an independent set of G of maximum weight $w(I) = \sum\limits_{i \in I} w_i$.

In [15], S. Sakai, M. Togasaki and K. Yamazaki gave two algorithms each providing a lower bound for $w(I)$:

$$w(I) \geq B_1(G) := \sum_{i \in V(G)} \frac{w_i}{d_G(i)+1} \tag{1.6}$$

and

$$w(I) \geq B_2(G) := \sum_{i \in V(G)} \sum_{k \in N_G(i) \cup \{i\}} w_k. \tag{1.7}$$

Both inequalities are generalizations of the Caro-Wei bound $\alpha(G) \geq \sum\limits_{i \in V(G)} \frac{1}{d_G(i)+1}$ ([3, 17]).

Using Statement 2, we define some lower bounds for $w(I)$, the maximum weight among independent sets of G. In some cases they are slightly better than these in (1.6) and (1.7) considered in [15].

Corollary 1.4.2

If $w_i > 0$ for $i \in V(G)$ and I is an independent set of G of maximum weight $w(I) = \sum\limits_{i \in I} w_i$, then with $x_i = \frac{w_i}{d_G(i)+1}$ we obtain by Statement 2:

$$w(I) \geq \frac{B_1(G)^2}{B_1(G) - \sum\limits_{ij \in E(G)} \left(\frac{1}{d_G(i)+1} - \frac{1}{d_G(j)+1}\right)\left(\frac{w_i}{d_G(i)+1} - \frac{w_j}{d_G(j)+1}\right)}. \tag{1.8}$$

PROOF: Set $x_i = \frac{w_i}{d_G(i)+1}$, then by Statement 2:

$$\begin{aligned} w(I) &\geq \frac{\left(\sum\limits_{i \in V(G)} \dfrac{w_i}{d_G(i)+1}\right)^2}{\sum\limits_{i \in V(G)} \frac{w_i}{(d_G(i)+1)^2} + \sum\limits_{ij \in E(G)} (\frac{1}{w_i} + \frac{1}{w_j}) \frac{w_i w_j}{(d_G(i)+1)(d_G(j)+1)}} \\ &= \frac{\left(\sum\limits_{i \in V(G)} \dfrac{w_i}{d_G(i)+1}\right)^2}{\sum\limits_{i \in V(G)} \left(\frac{w_i}{(d_G(i)+1)^2} + \frac{d_G(i) w_i}{(d_G(i)+1)^2}\right) + \sum\limits_{ij \in E(G)} \left((\frac{1}{w_i} + \frac{1}{w_j}) \frac{w_i w_j}{(d_G(i)+1)(d_G(j)+1)} - \frac{w_i}{(d_G(i)+1)^2} - \frac{w_j}{(d_G(j)+1)^2}\right)} \\ &= \frac{\left(\sum\limits_{i \in V(G)} \dfrac{w_i}{d_G(i)+1}\right)^2}{\sum\limits_{i \in V(G)} \frac{w_i}{d_G(i)+1} - \sum\limits_{ij \in E(G)} (\frac{1}{d_G(i)+1} - \frac{1}{d_G(j)+1})(\frac{w_i}{d_G(i)+1} - \frac{w_j}{d_G(j)+1})}. \end{aligned}$$

■

Remark 1.4.3 *Suppose the sum*

$$\sum_{ij \in E(G)} \left(\frac{1}{d_G(i)+1} - \frac{1}{d_G(j)+1}\right)\left(\frac{w_i}{d_G(i)+1} - \frac{w_j}{d_G(j)+1}\right) > 0,$$

is positive then Corollary 1.4.2 is stronger than inequality (1.6). For example, this is the case if G is non-regular and $\frac{d_G(i)+1}{d_G(j)+1} < \frac{w_i}{w_j}$ for all edges $ij \in E(G)$ with $d_G(i) < d_G(j)$.

Next, we investigate the other bound $B_2(G)$.

Corollary 1.4.4

If $w_i > 0$ for $i \in V(G)$ and I is an independent set of G of maximum weight $w(I) = \sum_{i \in I} w_i$, then with $x_i = \frac{w_i^2}{\sum_{k \in N(i)\cup\{i\}} w_k}$ we obtain by Statement 2:

$$w(I) \geq \frac{B_2(G)^2}{B_2(G) - S} \tag{1.9}$$

where $S := \sum_{i \in V(G)} \frac{w_i^2 \cdot \sum_{k \in N(i)} w_k}{\left(\sum_{k \in N(i)\cup\{i\}} w_k\right)^2} - \sum_{ij \in E(G)} \frac{w_i w_j (w_i + w_j)}{\left(\sum_{k \in N(i)\cup\{i\}} w_k\right)\left(\sum_{k \in N(j)\cup\{j\}} w_k\right)}.$

PROOF: Set $x_i = \frac{w_i^2}{\sum_{k \in N(i)\cup\{i\}} w_k}$, and proceed analogously as in the proof of Corollary 1.4.2. ∎

Remark 1.4.5 *If S in Corollary 1.4.4 is positive, then this bound for the weight of the independent set I is stronger than inequality (1.7).*

After considering an example for a bound on the independence number and two examples for a bound on the weight of a maximum weighted independent set, where an abundance of bounds exists, we turn to the more general cases. It has not yet been possible to us to find some results investigating bounds on the weight of induced subgraphs other than edgeless ones, referring to independence, or complete ones, which can be translated to a clique.

Even though we cannot evaluate the main results of bounds for the weight of general induced subgraphs, let us consider a bound on the weight of H for the case that H is an induced union of complete subgraphs as an example.

Example 1.4.6 *Let $w_i > 0$ for $i \in V(G)$, and let w and b denote the minima of the vertex and edge weights, respectively; i. e.*

$$w = \min_{i \in V(G)} w_i, \qquad \text{and} \qquad b = \min_{ij \in E(G)} \max\left\{\frac{1}{2w_i} + \frac{1}{w_j}, \frac{1}{w_i} + \frac{1}{2w_j}\right\}.$$

Then with $x_i = w_i$ there are $r \in \mathbb{N}$ and disjoint cliques $H_1, \dots, H_r$ of G such that

$$\sum_{s=1}^{r} w(H_s) - \frac{bw}{2 + bw} \cdot \sum_{\substack{s \in \{1,\dots,r\},\\ |V(H_s)| \geq 2}} w(H_s) \geq \frac{w(G)^2}{w(G) + \sum_{ij \in E(G)} \max\left\{\frac{1}{2}w_i + w_j, w_i + \frac{1}{2}w_j\right\}}.$$

PROOF: Set $x_i = w_i$ for $i \in V(G)$ and use Remark 1.3.6, which has been derived from Theorem 2. ∎

Note that the x_i have been set freely. More and possibly better bounds can be derived by distinct choices of x_i.

1.5 Algorithm

Let $w_i > 0$ for $i \in V(G)$, $b_{ij} \geq 0$ for $ij \in E(G)$ and $\mathcal{F} = \{F_1, \dots, F_p\}$ be a finite family of graphs with $K_1 \notin \mathcal{F}$. If $C(F_1), \dots, C(F_p)$ are fulfilled then there is for fixed $\mathcal{F}$ and fixed graph G a *polynomial time algorithm* expecting a vector x with $x_i \geq 0$ for $i \in V(G)$ and delivering an $\mathcal{F}$-free induced subgraph H of G fulfilling

$$w(H) \geq \frac{\left(\sum\limits_{i \in V(G)} x_i\right)^2}{\sum\limits_{i \in V(G)} \frac{x_i^2}{w_i} + \sum\limits_{ij \in E(G)} b_{ij} x_i x_j}. \tag{1.10}$$

Algorithm 1: Algorithm for a bound on $w(H)$

Input: Vector x with real $x_i \geq 0$ for $i \in V(G)$, satisfying $\sum_i x_i \neq 0$.
Output: An $\mathcal{F}$-free induced subgraph H of G fulfilling the inequality (1.10).

1 $\mathsf{x} \leftarrow x$; $\mathsf{H} \leftarrow$ `getH(x)`;
2 **while** *there is an induced subgraph F' of* H *isomorphic to an $F \in \mathcal{F}$* **do**
3     $\mathsf{x} \leftarrow$ `generateNewVector(x,`F'`)`;
4     $\mathsf{H} \leftarrow$ `getH(x)`;
5 **end**

The algorithm consists of two subroutines. The first one is to find an induced subgraph F' of H isomorphic to some $F \in \mathcal{F}$ in step 2; the second one is to get a new vector x such that the depending induced subgraph H does no longer contain F'. An explicit description of these steps can be found in the section 2.1.

Because $\mathcal{F}$ is a finite family of graphs, the first subroutine can be broken down into the following problem for each graph $F \in \mathcal{F}$: Find an induced subgraph F' of H isomorphic to the graph F.

The general case of this problem is chiefly solved by a brute-force-algorithm, for example see [16]. Choosing a set S of $|V(F)|$ vertices of H, the question is whether $H[S]$ is isomorphic to F. For more details on the isomorphism problem, we refer the interested reader to [5]. There are $\binom{|V(H)|}{|V(F)|} = \mathcal{O}\left(|V(H)|^{|V(F)|}\right)$ possible combinations for choosing S. The isomorphism problem is independent of $|V(H)|$; hence, it is solvable in $\mathcal{O}(1)$.

Note that for some small graphs on less or equal than four vertices, there are algorithms with a slightly better time complexity; for further details see [10]. Thus, step 2 in Algorithm 1 is solvable in at most $\mathcal{O}\left(|V(H)|^m\right)$, where $m = \max\limits_{F \in \mathcal{F}} |V(F)|$.

Steps 3 and 4 are solvable $\mathcal{O}(1)$. In each iteration the induced subgraph H is reduced by at least one vertex; thus, the loop will be executed at most $|V(H)|$ times. Therefore, an $\mathcal{F}$-free induced subgraph H of G fulfilling the inequality (2) can be constructed in $\mathcal{O}\left(|V(H)|^{m+1}\right)$. All these assertions will be proofed in section 2.1.

2 Proofs

Henceforth, let $G = (V, E)$ be a simple, finite, undirected graph with vertices $V(G)$ and edge set $E(G)$, and let U always be an arbitrary induced subgraph of G. Then S_U denotes the set of all real n-dimensional vectors x with entries x_i for $i \in V(G)$ such that $x_i = 0$ for $i \in V(G) \setminus V(U)$, $x_i \geq 0$ for $i \in V(U)$ and $\sum\limits_{i \in V(U)} x_i = 1$.

For given $x \in S_U$ let $H = H_x$ denote the induced subgraph of U obtained by deleting the vertices $i \in V(U)$ with $x_i = 0$, i. e. $H_x = U[\mathcal{X}_x]$ where $\mathcal{X}_x = \{i \in V(U) \ : \ x_i > 0\}$.

In [13] the discussion is extended to quadratic forms. They investigate

$$x \mapsto \sum_{i \in V(G)} d_G(i) x_i^2 + b \sum_{ij \in E(G)} x_i x_j,$$

where $b \in \mathbb{R}$ is a real number and $x \in S_G$. The maximum over S_G of this quadratic form is determined and partial results for the minimum are presented.

For the graph G let real numbers a_i for $i \in V(G)$ and b_{ij} for $ij \in E(G)$ be given. Henceforth, we identify $a_i = \frac{1}{w_i}$ for the vertex weights w_i already mentioned. The parameters b_{ij}, $ij \in E(G)$ keep their meaning from the first chapter. We will shortly talk about a's and b's instead of a_i for all $i \in V(G)$ and b_{ij} for all $ij \in E(G)$, respectively.

Motivated by the results in [13], we consider the quadratic form

$$\phi_U(x) = \sum_{i \in V(U)} a_i x_i^2 + \sum_{ij \in E(U)} b_{ij} x_i x_j \tag{2.1}$$

for an induced subgraph U of G.

Further, let

$$f(U) = \min_{x \in S_U} \phi_U(x) \tag{2.2}$$

and

$$g(U) = \max_{x \in S_U} \phi_U(x), \tag{2.3}$$

denote the minimum and maximum of ϕ_U over the set S_U, respectively. Note that S_U is a compact subset of $\mathbb{R}^n$, and ϕ_U is continuous; hence, f and g are well defined.

In section 2.1 we start with some general properties of the optimization problem. Thereafter, the optimal solutions $x \in S_G$ are investigated and the meaning for the depending subgraph H_x is exemplified. We conclude with the proof of Theorem 2. With these results we try to calculate

$f(G)$ for some special cases in section 2.2. This includes a generalization of Theorems 4 and 5 in T. S. Motzkin's and E. G. Straus's paper [13]. Finally, a proof of Theorem 1 is given.

2.1 Optimal solutions and the proof of Theorem 2

We start this section with some simple properties of ϕ_G and f. For this, consider the following lemma:

Lemma 2.1.1 (Properties of f)

a) Let T be an induced subgraph of U and $x \in S_T$. Then $x \in S_U$.

b) Let $x \in S_U$, then $\phi_G(x) = \phi_U(x)$.

c) Let $x \in S_U$. If $H_x = U[\mathcal{X}_x]$, then $\phi_G(x) = \phi_{H_x}(x)$.

d) If T is an induced subgraph of U, then $f(U) \leq f(T)$.

e) Denote by a the minimum of all a's, then $f(G) \leq a$.

f) Let $f(\cdot)$ also vary in the parameters a's and b's. Then $f(U)$ is continuous in all these parameters.

PROOF: In this proof let T be an induced subgraph of U.

a) For $x \in S_T$ we see $x_i = 0$ for all $i \in V(G) \setminus V(T)$; thus, $x_i = 0$ for all $i \in V(G) \setminus V(U)$; hence, $x \in S_U$.

b) For $x \in S_U$ it holds

$$\phi_G(x) = \sum_{i \in V(U)} a_i x_i^2 + \underbrace{\sum_{i \notin V(U)} a_i x_i^2}_{=0} + \sum_{ij \in E(U)} b_{ij} x_i x_j + \underbrace{\sum_{ij \in E(G) \setminus E(U)} b_{ij} x_i x_j}_{=0} = \phi_U(x),$$

because U is an induced subgraph of G.

c) For $x \in S_U$ let H_x again be the induced subgraph of G obtained by deleting the vertices $i \in V(G)$ with $x_i = 0$. Then $x \in S_{H_x}$, and b) proofs the assertion.

d) Let $x \in S_T$ be a minimal solution of ϕ_T, i. e. $\phi_T(x) = f(T)$, and $H_x = T[\mathcal{X}_x]$ be the induced subgraph of T. Then we obtain

$$f(T) = \phi_T(x) \overset{a), b)}{=} \phi_U(x) \geq f(U).$$

e) Let $a = \min_i a_i$, chose $i \in V(G)$ so that $a_i = a$, and let $x \in S_G$ with $x_i = 1$, and $x_j = 0$ for $j \neq i$ otherwise. Then $f(G) \leq \phi_G(x) = a_i = a$.

f) $f(G) = \min\limits_{x \in S_G} \phi_G(x)$ is a parametric optimization problem with constant feasible set. These problems are continuous in their parameters, so is $f(\cdot)$. A proof for this assertion can be found in [1, Theorems 1 and 2, pp. 115f]. ∎

Now we start the investigation of the optimization problems (2.2) and (2.3) by deriving a necessary condition for local optimality.

The main ideas of the proofs of the following Lemma 2.1.2 and Proposition 2.1.4 are already contained in [13].

Lemma 2.1.2 (Necessary condition for local optimality)

For U an induced subgraph of G let $x \in S_U$ be a locally optimal solution of ϕ_U and $H = H_x$ be the dependent induced subgraph.

Then the following equation holds for all $i \in V(H)$:

$$2a_i x_i + \sum_{j \in N_H(i)} b_{ij} x_j = 2\phi_U(x). \tag{2.4}$$

PROOF: By Lemma 2.1.1, case c), $\phi_U(x) = \phi_{H_x}(x)$. In this proof, we denote for $x \in \mathbb{R}^n_{\geq 0}$, $x \neq 0$, the vector $\tilde{x}$ that is obtained from x by deleting all components with $x_i = 0$. Then there is a $r_x \in \mathbb{N}$, $r_x \leq n$ such that $\tilde{x} \in \mathbb{R}^{r_x}_{>0}$.

For H_x consider the auxiliary form $\psi_{H_x} : \mathbb{R}^{r_x}_{\geq 0} \mapsto \mathbb{R}$

$$\psi_{H_x}(z) = \frac{\sum\limits_{i \in V(H_x)} a_i z_i^2 + \sum\limits_{ij \in E(H_x)} b_{ij} z_i z_j}{\left(\sum\limits_{i \in V(H_x)} z_i \right)^2}.$$

It is easy to be seen that $\psi_{H_x}(z) = \psi_{H_x}(\lambda z)$ for all $z \in \mathbb{R}^{r_x}_{\geq 0}$ and for all $\lambda > 0$.

If x is a local optimal solution of ϕ_U, then $\tilde{x}$ will be one of ψ_{H_x}, too. Moreover, $\tilde{x}$ is an interior point of the positive orthant of $\mathbb{R}^{r_x}$. Thus, all partial derivatives of ψ_{H_x} must be zero, and for all $i \in V(H_x)$ yield $\frac{\partial}{\partial \tilde{x}_i} \psi_{H_x}(\tilde{x}) = 0$

$$\overset{\text{Quotient rule}}{\Longleftrightarrow} \quad \left(2a_i x_i + \sum_{j \in N_{H_x}(i)} b_{ij} x_j \right) \underbrace{\left(\sum_{j \in V(H_x)} x_j \right)^2}_{=1} = \left(\sum_{j \in V(H_x)} a_j x_j^2 + \sum_{jk \in E(H_x)} b_{jk}\, x_j x_k \right) \cdot 2 \cdot \underbrace{\left(\sum_{j \in V(H_x)} x_j \right)}_{=1}$$

$$\Longleftrightarrow \quad 2a_i x_i + \sum_{j \in N_{H_x}(i)} b_{ij} x_j = 2\phi_{H_x}(x) = 2\phi_U(x),$$

which completes the proof. ∎

Using this condition of local optimality, we derive a lower bound for $f(U)$.

Corollary 2.1.3

Let U be an induced subgraph of G, and let $x \in S_U$ be a minimal solution of ϕ_U. $H = H_x$ denotes the depending subgraph. Suppose $a_i > 0$ for $i \in V(G)$ and $b_{ij} \geq 0$ for $ij \in E(G)$, and set $b = \min\limits_{ij \in E(U)} b_{ij}$ and $A = \max\limits_{i \in V(U)} a_i$.

Then the following bound can be derived:

$$\frac{1}{f(U)} \leq \sum_{i \in V(H)} \frac{1}{a_i} - \frac{b}{b + 2A} \sum_{\substack{i \in V(H), \\ d_H(i) \geq 1}} \frac{1}{a_i}. \tag{2.5}$$

If additionally H is edgeless then it follows

$$\frac{1}{f(U)} = \sum_{i \in V(H)} \frac{1}{a_i}.$$

Proof: $x \in S_U$ is a minimal solution of ϕ_U; thus, the necessary condition of optimality in Lemma 2.1.2 holds for all $i \in V(H)$:

$$2a_i x_i + \sum_{j \in N_H(i)} b_{ij} x_j = 2f(U).$$

For an isolated vertex $i \in V(H)$ the value x_i can be calculated by $x_i = \frac{f(U)}{a_i}$.

The vector $x \in S_H$ satisfies $\sum\limits_{i \in V(H)} x_i = 1$. It follows

$$\begin{aligned}
1 - \tfrac{b}{2A+b} &= \tfrac{2A}{2A+b} \sum_{i \in V(H)} x_i = \tfrac{2A}{2A+b} f(U) \sum_{\substack{i \in V(H),\\ d_H(i)=0}} \frac{1}{a_i} + \tfrac{2A}{2A+b} \sum_{\substack{i \in V(H),\\ d_H(i)\geq 1}} \left(\frac{f(U)}{a_i} - \sum_{j \in N_H(i)} \frac{b_{ij}}{2a_i} x_j \right) \\
&\leq \tfrac{2A}{2A+b} f(U) \sum_{i \in V(H)} \frac{1}{a_i} - \tfrac{2A}{2A+b} \tfrac{b}{2A} \sum_{\substack{i \in V(H)\\ d_H(i)\geq 1}} \sum_{j \in N_H(i)} x_j \\
&\leq \tfrac{2A}{2A+b} f(U) \sum_{i \in V(H)} \frac{1}{a_i} - \tfrac{b}{2A+b} \sum_{\substack{j \in V(H),\\ d_H(j)\geq 1}} \underbrace{d_H(j)}_{\geq 1} x_j \\
&\leq \tfrac{2A}{2A+b} f(U) \sum_{i \in V(H)} \frac{1}{a_i} - \tfrac{b}{2A+b} \left(1 - \sum_{\substack{j \in V(H),\\ d_H(j)=0}} x_j \right) \\
&= \tfrac{2A}{2A+b} f(U) \sum_{i \in V(H)} \frac{1}{a_i} - \tfrac{b}{2A+b} \left(1 - \sum_{i \in V(H)} \frac{f(U)}{a_i} + \sum_{\substack{j \in V(H),\\ d_H(j)\geq 1}} \frac{f(U)}{a_i} \right) \\
&= f(U) \sum_{i \in V(H)} \frac{1}{a_i} - \tfrac{b}{2A+b} f(U) \sum_{\substack{j \in V(H),\\ d_H(j)\geq 1}} \frac{1}{a_i} - \frac{b}{2A+b}.
\end{aligned}$$

If H is edgeless, then the second sums in all terms are omitted because there is no vertex $i \in V(H)$ with $d_H(i) \geq 1$, and equality holds everywhere. ∎

Next, we transfer the necessary condition for optimality to the condition $C(F)$ on H_x. Note that $a_i = \frac{1}{w_i}$ for $i \in V(G)$; thus, the condition onto the parameters a's and b's is denoted as follows:

Definition ($C(F)$)

Let F be an arbitrary graph, then the condition $C(F)$ is defined as follows:

For each induced subgraph U of G isomorphic to F there is a proper vertex partition of $V(U)$

into V_1 and V_2 fulfilling the inequality

$$\nu_2^2 \sum_{i\in V(U_1)} a_i + \nu_1^2 \sum_{i\in V(U_2)} a_i + \nu_2^2 \sum_{ij\in E(U_1)} b_{ij} + \nu_1^2 \sum_{ij\in E(U_2)} b_{ij} \le \nu_1\nu_2 \sum_{ij\in E^*} b_{ij}, \qquad (C(F))$$

where $U_1 = U[V_1]$ and $U_2 = U[V_2]$ denote the induced subgraphs on $\nu_1 \ge 1$ and $\nu_2 \ge 1$ vertices, respectively. $E^* = \{ij \in E(U) \,:\, i \in V_1, j \in V_2\}$ is the set of edges of U between V_1 and V_2.

The condition $C(F)$ leads to a condition for the optimal subgraph H_x.

Proposition 2.1.4 (Prohibited subgraphs of G)

Let $\mathcal{F}$ be a finite family of graphs. If $C(F)$ is fulfilled for all $F \in \mathcal{F}$, then there exists a minimal solution $x \in S_G$ of (2.2), i. e. $\phi_G(x) = f(G)$, such that $H_x = G[\mathcal{X}_x]$ is $\mathcal{F}$-free.

Proof: We chose an $x \in S_G$ such that $\phi_G(x) = f(G)$ and $|V(H_x)|$ is as small as possible. Assume H_x contains U isomorphic to $F \in \mathcal{F}$ as induced subgraph, then a new vector $\tilde{x}$ is obtained by increasing the value of x_i for $i \in V_1$ and decreasing it for $i \in V_2$ such that $\tilde{x}$ remains an element of the set S_G, where V_1 and V_2 are sets from a suitable proper vertex partition of $V(U)$.

Because $C(F)$ is fulfilled, there is a proper vertex partition of $V(U)$ into V_1 and V_2 satisfying the inequality

$$\nu_2^2 \sum_{i\in V(U_1)} a_i + \nu_1^2 \sum_{i\in V(U_2)} a_i + \nu_2^2 \sum_{ij\in E(U_1)} b_{ij} + \nu_1^2 \sum_{ij\in E(U_2)} b_{ij} \le \nu_1\nu_2 \sum_{ij\in E^*} b_{ij}.$$

Choose $\varepsilon > 0$ such that $\tilde{x}_i := x_i + \frac{\varepsilon}{\nu_1} \le 1$ for all $i \in V(U_1)$ and $0 \le \tilde{x}_i := x_i - \frac{\varepsilon}{\nu_2}$ for all $i \in V(U_2)$. Additionally set $\tilde{x}_i := x_i$ for all $i \notin V(U_1) \cup V(U_2)$. We obtain $\tilde{x} \in S_G$.

Now, we calculate $\phi_G(\tilde{x})$ and transform it to $\phi_G(x) + R$ with a residual R. According to $\phi_G(x) = f(G)$, R has some restrictions leading to a contradiction to the assumption that H_x contains U:

$$\begin{aligned}
f(G) &= \phi_{H_x}(x) \\
&\le \phi_{H_x}(\tilde{x}) \\
&= \sum_{\substack{i\,\in\,V(H),\\ i\,\notin\,V(U)}} a_i x_i^2 + \sum_{i\,\in\,V(U_1)} a_i (x_i + \tfrac{\varepsilon}{\nu_1})^2 + \sum_{i\,\in\,V(U_2)} a_i (x_i - \tfrac{\varepsilon}{\nu_2})^2 \\
&\quad + \sum_{\substack{ij\,\in\,E(H),\\ i,j\,\notin\,V(U)}} b_{ij} x_i x_j + \sum_{\substack{ij\,\in\,E(H),\\ i\,\in\,V(U_1),\\ j\,\notin\,V(U)}} b_{ij} (x_i + \tfrac{\varepsilon}{\nu_1}) x_j \\
&\quad + \sum_{\substack{ij\,\in\,E(H),\\ i\,\in\,V(U_2),\\ j\,\notin\,V(U)}} b_{ij} (x_i - \tfrac{\varepsilon}{\nu_2}) x_j + \sum_{\substack{ij\,\in\,E(H),\\ i,j\,\in\,V(U_1)}} b_{ij} (x_i + \tfrac{\varepsilon}{\nu_1})(x_j + \tfrac{\varepsilon}{\nu_1}) \\
&\quad + \sum_{\substack{ij\,\in\,E(H),\\ i,j\,\in\,V(U_2)}} b_{ij} (x_i - \tfrac{\varepsilon}{\nu_2})(x_j - \tfrac{\varepsilon}{\nu_2}) + \sum_{\substack{ij\,\in\,E(H),\\ i\,\in\,V(U_1),\\ j\,\in\,V(U_2)}} b_{ij} (x_i + \tfrac{\varepsilon}{\nu_1})(x_j - \tfrac{\varepsilon}{\nu_2})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\,\in\,V(H)} a_i x_i^2 + 2\tfrac{\varepsilon}{\nu_1} \sum_{i\,\in\,V(U_1)} a_i x_i + (\tfrac{\varepsilon}{\nu_1})^2 \sum_{i\,\in\,V(U_1)} a_i - 2\tfrac{\varepsilon}{\nu_2} \sum_{i\,\in\,V(U_2)} a_i x_i + (\tfrac{\varepsilon}{\nu_2})^2 \sum_{i\,\in\,V(U_2)} a_i \\
&\quad + \sum_{ij\,\in\,E(H)} b_{ij} x_i x_j + \sum_{\substack{ij\,\in\,E(H),\\ i\,\in\,V(U_1),\\ j\,\notin\,V(U)}} b_{ij} \tfrac{\varepsilon}{\nu_1} x_j - \sum_{\substack{ij\,\in\,E(H),\\ i\,\in\,V(U_2),\\ j\,\notin\,V(U)}} b_{ij} \tfrac{\varepsilon}{\nu_2} x_j \\
&\quad + \sum_{\substack{ij\,\in\,E(H),\\ i,j\,\in\,V(U_1)}} b_{ij} \tfrac{\varepsilon}{\nu_1}(x_i + x_j) - \sum_{\substack{ij\,\in\,E(H),\\ i,j\,\in\,V(U_2)}} b_{ij} \tfrac{\varepsilon}{\nu_2}(x_i + x_j) + (\tfrac{\varepsilon}{\nu_1})^2 \sum_{\substack{ij\,\in\,E(H),\\ i,j\,\in\,V(U_1)}} b_{ij} + (\tfrac{\varepsilon}{\nu_2})^2 \sum_{\substack{ij\,\in\,E(H),\\ i,j\,\in\,V(U_2)}} b_{ij} \\
&\quad + \sum_{\substack{ij\,\in\,E(H),\\ i\,\in\,V(U_1),\\ j\,\in\,V(U_2)}} b_{ij} \tfrac{\varepsilon}{\nu_1} x_j - \sum_{\substack{ij\,\in\,E(H),\\ i\,\in\,V(U_1),\\ j\,\in\,V(U_2)}} b_{ij} \tfrac{\varepsilon}{\nu_2} x_i - \tfrac{\varepsilon}{\nu_1}\tfrac{\varepsilon}{\nu_2} \sum_{\substack{ij\,\in\,E(H),\\ i\,\in\,V(U_1),\\ j\,\in\,V(U_2)}} b_{ij} \\
&= f(G) + (\tfrac{\varepsilon}{\nu_1})^2 \sum_{i\,\in\,V(U_1)} a_i + (\tfrac{\varepsilon}{\nu_2})^2 \sum_{i\,\in\,V(U_2)} a_i + (\tfrac{\varepsilon}{\nu_1})^2 \sum_{\substack{ij\,\in\,E(H),\\ i,j\,\in\,V(U_1)}} b_{ij} + (\tfrac{\varepsilon}{\nu_2})^2 \sum_{\substack{ij\,\in\,E(H),\\ i,j\,\in\,V(U_2)}} b_{ij} - \tfrac{\varepsilon}{\nu_1}\tfrac{\varepsilon}{\nu_2} \sum_{\substack{ij\,\in\,E(H),\\ i\,\in\,V(U_1),\\ j\,\in\,V(U_2)}} b_{ij} \\
&\quad + \tfrac{\varepsilon}{\nu_1} \sum_{i\,\in\,V(U_1)} \underbrace{\Big(2a_i x_i + \sum_{j\,\in\,N_H(i)} b_{ij} x_j\Big)}_{=2\phi_G(x),\text{ by Lemma 2.1.2}} - \tfrac{\varepsilon}{\nu_2} \sum_{i\,\in\,V(U_2)} \underbrace{\Big(2a_i x_i + \sum_{j\,\in\,N_H(i)} b_{ij} x_j\Big)}_{=2\phi_G(x),\text{ by Lemma 2.1.2}} \\
&= f(G) + \varepsilon^2 \Bigg(\tfrac{1}{\nu_1^2} \sum_{i\,\in\,V(U_1)} a_i + \tfrac{1}{\nu_2^2} \sum_{i\,\in\,V(U_2)} a_i + \tfrac{1}{\nu_1^2} \sum_{ij\,\in\,E(U_1)} b_{ij} + \tfrac{1}{\nu_2^2} \sum_{ij\,\in\,E(U_2)} b_{ij} - \tfrac{1}{\nu_1 \nu_2} \sum_{ij\,\in\,E^*} b_{ij} \Bigg) \\
&\quad + \tfrac{\varepsilon}{\nu_1} \nu_1 2\phi_G(x) - \tfrac{\varepsilon}{\nu_2} \nu_2 2\phi_G(x) \\
&= f(G) + \tfrac{\varepsilon^2}{\nu_1^2 \nu_2^2} \Bigg(\nu_2^2 \sum_{i\,\in\,V(U_1)} a_i + \nu_1^2 \sum_{i\,\in\,V(U_2)} a_i + \nu_2^2 \sum_{ij\,\in\,E(U_1)} b_{ij} + \nu_1^2 \sum_{ij\,\in\,E(U_2)} b_{ij} - \nu_1 \nu_2 \sum_{ij\,\in\,E^*} b_{ij} \Bigg).
\end{aligned}
$$

$C(F)$ is fulfilled and x is a minimal solution. Thus

$$
\nu_2^2 \sum_{i \in V(U_1)} a_i + \nu_1^2 \sum_{i \in V(U_2)} a_i + \nu_2^2 \sum_{ij \in E(U_1)} b_{ij} + \nu_1^2 \sum_{ij \in E(U_2)} b_{ij} - \nu_1 \nu_2 \sum_{ij \in E^*} b_{ij} = 0,
$$

and $\tilde{x}$ is also a minimal solution for all ε making sense.

Suppose ε is chosen maximal, then there is at least one $i \in V(U_1)$ with $\tilde{x}_i := x_i + \frac{\varepsilon}{\nu_1} = 1$, i. e. $|V(H_{\tilde{x}})| = 1$, or an $i \in V(U_2)$ with $\tilde{x}_i := x_i - \frac{\varepsilon}{\nu_2} = 0$, i. e. $i \notin V(H_{\tilde{x}})$. In both cases $|V(H_{\tilde{x}})| < |V(H_x)|$, a contradiction to the choice of x, because $|V(H_x)|$ has not been chosen to be as small as possible; thus, H_x must be $\mathcal{F}$-free. ∎

Remark 2.1.5 *We can analogously proof the proceeding lemma for the case that $x \in S_G$ is a maximal solution of $\phi_G(x)$:*

Let F be an arbitrary graph. For each induced subgraph U of G isomorphic to F there is a certain proper vertex partition of $V(U)$ into V_1 and V_2. Let $U_1 = U[V_1]$ and $U_2 = U[V_2]$ denote the induced subgraphs on $\nu_1 \geq 1$ and $\nu_2 \geq 1$ vertices, respectively. Let E^* be the set of edges of U between V_1 and V_2, i. e. $E^* = \{ij \in E(U) \;:\; i \in V_1, j \in V_2\}$.

If for such a vertex partition of $V(U)$ the inequality

$$
\nu_2^2 \sum_{i \in V(U_1)} a_i + \nu_1^2 \sum_{i \in V(U_2)} a_i + \nu_2^2 \sum_{ij \in E(U_1)} b_{ij} + \nu_1^2 \sum_{ij \in E(U_2)} b_{ij} \geq \nu_1 \nu_2 \sum_{ij \in E^*} b_{ij} \tag{2.6}
$$

holds, then there exists a maximal solution $x \in S_G$ of (2.3), i. e. $\phi_G(x) = g(G)$, such that H_x is F-free. Note the reversed inequality sign in (2.6).

Remark 2.1.6 *If we suppose that $x \in S_G$ is a* maximal *solution of ϕ_G, i. e. $\phi_G(x) = g(G)$, then by Remark 2.1.5 the inequality (2.6) differs from the condition $C(F)$ only by the reversed inequality sign, which can be changed back by inverting the signs of all parameters a's and b's.*

Note that $g(G) = -\min_{x \in S_G} \sum_{i \in V(G)} (-a_i)x_i^2 + \sum_{ij \in E(G)} (-b_{ij})x_i x_j$.

Thus, we can replace the optimization problem $g(G)$ in (2.3) by minimizing ϕ_G as in (2.2). For this, we only have to invert the signs of all parameters a's and b's.

Therefore, the investigation of the minimum and $f(G)$ will be sufficient, and we may restrict to that case.

At this point, the proof of Theorem 2 is given. For the convenience of the reader the theorem from page 3 is recapitulated below.

Theorem 2

Let $\mathcal{F} = \{F_1, \dots, F_p\}$ be a finite family of graphs with $K_1 \notin \mathcal{F}$, and let $w_i > 0$ for $i \in V(G)$ and $b_{ij} \geq 0$ for $ij \in E(G)$ be the weights on the vertices and egdes. Denote the minima over the edge and vertex weights by $b = \min_{ij \in E(G)} b_{ij}$ and $w = \min_{i \in V(G)} w_i$, respectively.

If $C(F_1), \dots, C(F_p)$ are fulfilled, then there is an $\mathcal{F}$-free induced subgraph H of G such that

$$w(H) - \frac{bw}{2 + bw} \cdot \sum_{\substack{i \in V(H),\\ d_H(i) \geq 1}} w_i \geq \frac{\left(\sum_{i \in V(G)} x_i \right)^2}{\sum_{i \in V(G)} \frac{x_i^2}{w_i} + \sum_{ij \in E(G)} b_{ij} x_i x_j}$$

for all real $x_i \geq 0$ with $i \in V(G)$, satisfying $\sum_i x_i \neq 0$.

Proof: (of Theorem 2) Let $x \in S_G$ be a minimal solution of (2.2), i. e. $\phi_G(x) = f(G)$, we considered $H = G[\mathcal{X}_x]$ as the subgraph of G obtained by deleting the vertices $i \in V(G)$ with $x_i = 0$.

The conditions $C(F_1), \dots, C(F_p)$ for a finite family $\mathcal{F} = \{F_1, \dots, F_p\}$ are fulfilled; thus, H can be chosen to be a $\mathcal{F}$-free induced subgraph of G by Proposition 2.1.4.

Finally with Corollary 2.1.3 we obtain for all $z \in \mathbb{R}^n_{\geq 0}$ satisfying $z \neq 0$:

$$\begin{aligned}
w(H) - \frac{bw}{2 + bw} \sum_{\substack{i \in V(H),\\ d_H(i) \geq 1}} w_i &= \sum_{i \in V(H)} \frac{1}{a_i} - \frac{b}{b + 2 \max_{i \in V(G)} a_i} \sum_{\substack{i \in V(H),\\ d_H(i) \geq 1}} \frac{1}{a_i} \\
&\geq \frac{1}{f(G)} = \frac{1}{\phi_G(x)} \\
&\geq \frac{1}{\phi_G(y)} = \frac{\left(\sum_{i \in V(G)} z_i \right)^2}{\sum_{i \in V(G)} \frac{z_i^2}{w_i} + \sum_{ij \in E(G)} b_{ij} z_i z_j},
\end{aligned}$$

where y is the normalised vector z, i. e. $y := \frac{z}{\sum_i z_i} \in S_G$. H_x is the $\mathcal{F}$-free induced subgraph of G satisfying the assumption. ∎

Now, we investigate the Algorithm 1 from section 1.5:

Algorithm 1: Algorithm for a bound on $w(H)$

Input: Vector x with real $x_i \geq 0$ for $i \in V(G)$, satisfying $\sum_i x_i \neq 0$.
Output: An $\mathcal{F}$-free induced subgraph H of G fulfilling the inequality (1.10).

1 $\mathsf{x} \leftarrow x$; $\mathsf{H} \leftarrow$ `getH(x)`;
2 **while** *there is an induced subgraph* F' *of* H *isomorphic to an* $F \in \mathcal{F}$ **do**
3 $\quad \mathsf{x} \leftarrow$ `generateNewVector(x,`F'`)`;
4 $\quad \mathsf{H} \leftarrow$ `getH(x)`;
5 **end**

Step 2 has already been discussed; the remaining steps 1, 3, and 4 will be explained in the following.

Algorithm 2: Function constructs H

Function `getH(x)`:
$\quad \mathsf{H} \leftarrow H_{\mathsf{x}}$ that is the induced subgraph of G obtained by deleting the vertices $i \in V(G)$ with $\mathsf{x}_i = 0$, i. e. $H_{\mathsf{x}} = G[\mathcal{X}_{\mathsf{x}}]$ where $\mathcal{X}_{\mathsf{x}} = \{i \in V(G) \ : \ \mathsf{x}_i > 0\}$.
end

Algorithm 3: Function constructs a new vector x

Input: An induced subgraph F' of H isomorphic to a $F \in \mathcal{F}$.

Function `generateNewVector(x,`F'`)`:
$\quad$ According to $C(F)$ there is a proper vertex partition such that the inequality is fulfilled. Suppose that U_1 and U_2 are indexed such that

$$\frac{1}{\nu_1} \sum_{i \in V(U_1)} \Big(2a_i x_i + \sum_{j \in N_H(i)} b_{ij} x_j \Big) \leq \frac{1}{\nu_2} \sum_{i \in V(U_2)} \Big(2a_i x_i + \sum_{j \in N_H(i)} b_{ij} x_j \Big).$$

$\quad$ Choose $\varepsilon > 0$ maximal such that $0 \leq \tilde{x}_i := x_i - \frac{\varepsilon}{\nu_2}$ for all $i \in V(U_2)$. Additionally set $\tilde{x}_i := x_i + \frac{\varepsilon}{\nu_1} \leq 1$ for all $i \in V(U_1)$ and $\tilde{x}_i := x_i$ for all $i \notin V(U_1) \cup V(U_2)$.
$\quad \mathsf{x} \leftarrow \tilde{x}$
end

Proposition 2.1.7

Suppose that Algorithm 1 generates the sequences $x^{(k)}$ and $H^{(k)}$ with $k = 0, 1, 2, \dots$

It follows:

a) $V(H^{(k+1)}) \subsetneq V(H^{(k)})$ for all k,

b) The algorithm terminates; thus, there is a $r \in \mathbb{N}_0$ such that $k = 0, 1, \dots, r$,

c) $\phi_G(y^{(k+1)}) \geq \phi_G(y^{(k)})$ for all $k = 0, 1, \dots, r$, where $y_i = \frac{x_i}{\sum x_j}$ for $i \in V(G)$,

d) The algorithm outputs an $\mathcal{F}$-free induced subgraph H of G fulfilling the inequality (1.10).

Proof: Note that the condition $C(F)$ is fulfilled and symmetric. Thus, a suitable indexing is possible in step 2 of Algorithm 3.

a) We chose $\varepsilon > 0$ maximal, i. e. there is an $i \in V(U_2)$ such that $\tilde{x}_i = 0$. Note that $\tilde{x}_j = 0$ for all $j \in V(G)$ with $x_j = 0$. Thus, the new subgraph $H^{(k+1)}$ has at least one vertex less, and it yields $V(H^{(k+1)}) \subsetneq V(H^{(k)})$.

b) According to a), $|V(H^{(k+1)})| < |V(H^{(k)})|$ for all k; thus, the loop condition in step 2 of Algorithm 1 cannot always be true, and Algorithm 1 will terminate.

c) Let us again consider the proof of Proposition 2.1.4, it follows

$$\begin{aligned}\phi_{H_y}(\tilde{y}) = \phi_{H_y}(y) + \varepsilon^2 &\underbrace{\left(\tfrac{1}{\nu_1^2}\sum_{i\in V(U_1)} a_i + \tfrac{1}{\nu_2^2}\sum_{i\in V(U_2)} a_i + \tfrac{1}{\nu_1^2}\sum_{ij\in E(U_1)} b_{ij} + \tfrac{1}{\nu_2^2}\sum_{ij\in E(U_2)} b_{ij} - \tfrac{1}{\nu_1\nu_2}\sum_{ij\in E^*} b_{ij}\right)}_{\leq 0,\text{ because } C(F) \text{ is fulfilled,}} \\ &+ \underbrace{\tfrac{\varepsilon}{\nu_1}\sum_{i\in V(U_1)}\Big(2a_iy_i + \sum_{j\in N_H(i)} b_{ij}y_j\Big) - \tfrac{\varepsilon}{\nu_2}\sum_{i\in V(U_2)}\Big(2a_iy_i + \sum_{j\in N_H(i)} b_{ij}y_j\Big)}_{\leq 0,\text{ because of step 2 in Algorithm 3}} \\ \leq \phi_{H_y}(y).\end{aligned}$$

Note that $\tilde{y}_i = \frac{\tilde{x}_i}{\sum x_j}$ for all $i \in V(G)$ and $\sum_j x_j = \sum_j \tilde{x}_j$. Thus, $\phi_G(y^{(k+1)}) \geq \phi_G(y^{(k)})$ for all $k = 0, 1, \dots, r$.

d) Suppose Algorithm 1 terminates after $r \in \mathbb{N}_0$ loop passings. Let $y_i^{(r)} = \frac{x_i^{(r)}}{\sum x_j^{(r)}}$ for $i \in V(G)$ be the normalised vector of $x^{(r)}$, then it follows

$$\begin{aligned}\frac{\left(\sum\limits_{i\in V(G)} x_i^{(0)}\right)^2}{\sum\limits_{i\in V(G)} \frac{\left(x_i^{(0)}\right)^2}{w_i} + \sum\limits_{ij\in E(G)} b_{ij}x_i^{(0)}x_j^{(0)}} &= \frac{1}{\phi_G(y^{(0)})} \overset{\text{c)}}{\leq} \frac{1}{\phi_G(y^{(r)}))} \overset{\text{Lemma 2.1.1, c)}}{=} \frac{1}{\phi_{H^{(r)}}(y^{(r)})} \\ &\leq \frac{1}{f(H^{(r)})} \overset{\text{Corollary 2.1.3}}{\leq} \sum_{i\in V(\tilde{H})} \frac{1}{a_i} - \frac{b}{b+2A} \sum_{\substack{i\in V(\tilde{H}),\\ d_{\tilde{H}}(i)\geq 1}} \frac{1}{a_i} \\ &\leq \sum_{i\in V(\tilde{H})} \frac{1}{a_i} \leq \sum_{i\in V(H^{(r)})} \frac{1}{a_i},\end{aligned}$$

where $\tilde{H}$ denotes the induced subgraph obtained from an optimal solution of $f(H)$. ■

Remark 2.1.8 *Step 3 and 4 in Algorithm 1 are $\mathcal{O}(1)$ because all calculations depend on $|V(F')|$, not on $|V(G)|$.*

2.2 Minimum value and the proof of Theorem 1

After considering the optimal solutions of the optimization problem $f(G)$, we will investigate the value of $f(G)$ in this section, and describe it in a few propositions. The first two Propositions 2.2.1 and 2.2.2 have already been contained in [13] in a modified way. For cases where H consists of multiple components, a lemma indicates how to solve the problem. Finally, the proof of Theorem 1 is given.

For the following propositions let $x \in S_G$ be a minimal solution of (2.2), i.e. $\phi_G(x) = f(G)$. Again, $H = H_x = G[\mathcal{X}_x]$ denotes the subgraph of G obtained by deleting the vertices $i \in V(G)$ with $x_i = 0$.

Proposition 2.2.1 (Independent sets)

Let $x \in S_G$ be a minimal solution of ϕ_G, and $H = H_x$ the depending subgraph. Denote $a = \min_i a_i$, and suppose H is an edgeless graph.

a) If $a \leq 0$, then $f(G) = a$, and this minimum is attained by setting $x_i = 1$ for one vertex $i \in V(G)$ with $a_i = a$ and $x_j = 0$ for $j \neq i$. If additionally $a = 0$, the minimum can also be attained by freely choosing $x \in S_G$ fulfilling $x_i = 0$ for all $i \in V(G)$ not belonging to an independent set all of whose vertices j have $a_j = 0$.

b) If $a > 0$, then

$$f(G) = \frac{1}{\max\left(\sum_{i \in S} \frac{1}{a_i}\right)}, \tag{2.7}$$

where the maximum is taken over all non-empty independent sets S of G. The minimum in $f(G)$ is attained by setting $x_i = \frac{f(G)}{a_i}$ for i in the optimal independent set S, and $x_i = 0$ otherwise.

Proof: The necessary condition of optimality in Lemma 2.1.2 yields

$$2a_i x_i + \sum_{j \in N_H(i)} b_{ij} x_j = 2f(G) \tag{2.8}$$

for all $i \in V(H)$. H is edgeless; thus, there are no neighbours of i in H, and it follows

$$\sum_{j \in N_H(i)} b_{ij} x_j = 0.$$

We get

$$f(G) = a_i x_i, \qquad \text{for all } i \in V(H). \tag{2.9}$$

a) Let $x \in S_G$ such that ϕ_G attains its minimum. It follows

$$f(G) = \phi_H(x) = \sum_{i \in V(H)} a_i x_i^2 + \underbrace{\sum_{ij \in E(H)} b_{ij} x_i x_j}_{=0} \geq a \sum_{i \in V(H)} x_i^2 \quad \underbrace{+2a \sum_{\substack{i,j \in V(H) \\ i<j}} x_i x_j}_{\leq 0, \text{ because } a \leq 0} = a \cdot \left(\sum_{i \in V(H)} x_i \right)^2 = a.$$

Lemma 2.1.1, case e) yields $f(G) = a$. This minimum is attained by choosing $i \in V(G)$ so that $a_i = a$, and let $x \in S_G$ with $x_i = 1$ and $x_j = 0$ for $j \neq i$.

If $f(G) = a = 0$, then equation (2.9) is zero, and it follows $a_i = 0$ for $i \in V(H)$. The choice of x_i for $i \in V(H)$ is arbitrary.

b) Note that $a_i \geq a > 0$, and (2.9) is positive. Because of $x \in S_G$, setting $x_i = \frac{f(G)}{a_i}$ for all $i \in V(H)$ yields

$$1 = \sum_{i \in V(H)} x_i = f(G) \sum_{i \in V(H)} \frac{1}{a_i},$$

which completes the proof. ■

The results of the previous Proposition 2.2.1 for the case that $a_i = d_G(i)$ for all $i \in V(G)$ are partially presented in [13, Theorem 5].

For the next case let H be a complete graph. By Table 1.3.9, $a_i < 0$ suffices to force H being complete. Note that minimizing with negative a's is the same as maximizing ϕ_G with positive a's. Thus, the following Proposition 2.2.2 is a generalization of [13, Theorem 4], where the maximum is already completely discussed in case that $a_i = d_G(i)$ for $i \in V(G)$.

Proposition 2.2.2 (Generalised result of T. S. Motzkin and E. G. Straus [13])

Let $x \in S_G$ be a minimal solution of ϕ_G, and $H = H_x$ the depending subgraph. Denote $a = \min_{i \in V(G)} a_i$ and $b = \min_{ij \in E(G)} b_{ij}$, and suppose H is a complete graph.

a) If $2a \leq b$, then $f(G) = a$, and this minimum is attained by setting $x_i = 1$ for one vertex $i \in V(G)$ with $a_i = a$ and $x_j = 0$ for $j \neq i$.

b) If $2a > b$, then

$$f(G) \geq \frac{1}{2} \left(b + \frac{1}{\max \sum_{i \in C} \frac{1}{2a_i - b}} \right) \tag{2.10}$$

and the maximum is taken over all cliques C of G.

c) If additionally $b = b_{ij}$ for all $ij \in E(G)$, then equality holds in (2.10), and the minimum is attained by setting $x_i = \frac{2f(G)-b}{2a_i-b}$ for i in the optimal clique C and $x_i = 0$ otherwise.

Proof: The necessary condition of optimality in Lemma 2.1.2 yields

$$2a_i x_i + \sum_{j \in N_H(i)} b_{ij} x_j = 2f(G) \tag{2.11}$$

for all $i \in V(H)$. H is complete; thus, the neighbourhood of i is $V(H) \setminus \{i\}$, and it follows

$$\sum_{j \in N_H(i)} x_j = 1 - x_i.$$

We obtain

$$2a_i x_i + b(1 - x_i) \leq 2a_i x_i + \sum_{j \in N_H(i)} b_{ij} x_j = 2f(G);$$

hence, it follows

$$(2a_i - b)x_i \leq 2f(G) - b, \qquad \text{for all } i \in V(H). \tag{2.12}$$

a) Let $x \in S_G$ such that ϕ_G attains its minimum. It follows

$$f(G) = \phi_H(x) = \sum_{i \in V(H)} a_i x_i^2 + \sum_{ij \in E(H)} b_{ij} x_i x_j \geq a \sum_{i \in V(H)} x_i^2 + 2a \sum_{ij \in E(H)} x_i x_j = a \cdot \left(\sum_{i \in V(H)} x_i \right)^2 = a,$$

because H is a complete graph.

Lemma 2.1.1, case e) yields $f(G) = a$. This minimum is attained by choosing $i \in V(G)$ so that $a_i = a$, and let $x \in S_G$ with $x_i = 1$ and $x_j = 0$ for $j \neq i$.

b) Note that $2a_i \geq 2a > b$, and both terms in (2.12) are positive. Because of $x \in S_G$ and by (2.12), the estimation $x_i \leq \frac{2f(G)-b}{2a_i-b}$ for all $i \in V(H)$ yields

$$1 = \sum_{i \in V(H)} x_i \leq (2f(G) - b)) \sum_{i \in V(H)} \tfrac{1}{2a_i - b}.$$

This implies

$$f(G) \geq \frac{1}{2} \left(b + \frac{1}{\sum\limits_{i \in V(H)} \frac{1}{2a_i - b}} \right),$$

and the sum has to be maximal.

c) If $b = b_{ij}$ for all $ij \in E(G)$, then equality holds in (2.12), and therefore equality also holds in all inequalities in the proof of b). ∎

After considering the simple cases that H is complete or edgeless, further investigations lead to the question of how to proceed if H consists of multiple components. For this, the following lemma will give an answer.

Lemma 2.2.3 (Splitting lemma for components of H)

Suppose $a_i > 0$ for $i \in V(G)$ and $b_{ij} \geq 0$ for $ij \in E(G)$.

Let $x \in S_G$ be a minimal solution of (2.2), i. e. $\phi_G(x) = f(G)$. $H = H_x$ denotes the subgraph of G obtained by deleting the vertices $i \in V(G)$ with $x_i = 0$.

If H consist of components $H_1, \dots, H_r$, then

$$\frac{1}{f(G)} = \frac{1}{f(H_1)} + \cdots + \frac{1}{f(H_r)}.$$

PROOF: For $x \in S_G$ such that ϕ_G attains its minimum, let $z_s = \sum_{j \in V(H_s)} x_j > 0$ for $s = 1, \dots, r$, and substitute $y_i = \frac{x_i}{z_s} > 0$ for $i \in V(H_s)$.

Note that $z_1 + \cdots + z_r = 1$ and $\sum_{j \in V(H_s)} y_j = 1$ for $s = 1, \dots, r$.

It follows

$$\begin{aligned}
f(G) = \phi_G(x) &\overset{\text{Lemma 2.1.1 c)}}{=} \phi_H(x) \\
&= \sum_{s=1}^{r} \left(\sum_{i \in V(H_s)} a_i x_i^2 + \sum_{ij \in E(H_s)} b_{ij} x_i x_j \right) = \sum_{s=1}^{r} \left(z_s^2 \left(\sum_{i \in V(H_s)} a_i y_i^2 + \sum_{ij \in E(H_s)} b_{ij} y_i y_j \right) \right) \\
&= \sum_{s=1}^{r} \left(z_s^2 \phi_{H_s}(x) \right) \geq \sum_{s=1}^{r} z_s^2 f(H_s) \\
&\geq \min_{c \in S_r} \sum_{s=1}^{r} c_s^2 f(H_s).
\end{aligned}$$

Now let $z = (z_1, \dots, z_r) \in S_r$ be a minimal solution of $\min_{c \in S_r} \sum_{s=1}^{r} c_s^2 f(H_s)$, and for $s = 1, \dots, r$ let $f(H_s) = \phi_{H_s}\left(y^{(s)}\right)$ for an arbitrary $y^{(s)} \in S_{n_s}$.

Let $w_i = z_s y_i^{(s)}$ for $i \in V(H_s)$, and $w_i = 0$ otherwise. It is easy to be seen that $w \in S_G$. Then it yields:

$$\begin{aligned}
\min_{c \in S_r} \sum_{s=1}^{r} c_s^2 f(H_s) &= \sum_{s=1}^{r} z_s^2 \phi_{H_s}\left(y^{(s)}\right) \\
&= \sum_{s=1}^{r} \left(\sum_{i \in V(H_s)} a_i (z_s y_i^{(s)})^2 + \sum_{ij \in E(H_s)} b_{ij} (z_s y_i^{(s)})(z_s y_j^{(s)}) \right) \\
&= \sum_{s=1}^{r} \left(\sum_{i \in V(H_s)} a_i w_i^2 + \sum_{ij \in E(H_s)} b_{ij} w_i w_j \right) \\
&= \phi_H(w) \geq f(H) \overset{\text{Lemma 2.1.1 d)}}{\geq} f(G).
\end{aligned}$$

Hence, we obtain $f(G) = \min_{c \in S_r} \sum_{s=1}^{r} c_s^2 f(H_s)$.

Note that $f(H_s) > 0$ for $s = 1, \dots, r$ because all the parameters are positive.

We consider the relaxed problem

$$\min\left\{\sum_{s=1}^{r} c_s^2 f(H_s) \mid c_1+\cdots+c_r=1\right\}$$

and the Lagrange function

$$L=\sum_{s=1}^{r} c_s^2 f(H_s)+2\rho\left(1-\sum_{s=1}^{r} c_s\right)$$

with Lagrange multiplier $2\rho \in \mathbb{R}$.

The necessary condition of optimality is $\nabla_{c_s} L = 2c_s f(H_s) - 2\rho = 0$ for $s=1,\dots,r$. Thus, it follows $c_s = \frac{\rho}{f(H_s)}$, and with $c_1+\cdots+c_r=1$, we obtain

$$\frac{1}{\rho}=\frac{1}{\rho}\sum_{s=1}^{r} c_s=\sum_{s=1}^{r}\frac{1}{f(H_s)}>0,$$

so $\rho>0$, and thereby $c_s>0$ for $s=1,\dots,r$.

It follows:

$$\begin{aligned} f(G) &= \sum_{s=1}^{r} c_s^2 f(H_s) = \sum_{s=1}^{r}\frac{\rho^2}{f(H_s)^2} f(H_s) = \rho^2\sum_{s=1}^{r}\frac{1}{f(H_s)} \\ &= \left(\sum_{s=1}^{r}\frac{1}{f(H_s)}\right)^{-2}\sum_{s=1}^{r}\frac{1}{f(H_s)} = \left(\sum_{s=1}^{r}\frac{1}{f(H_s)}\right)^{-1}, \end{aligned}$$

which completes the proof. ■

Next, we use the proceeding lemma for the case that $C(P_3)$ holds. For this case, H can be chosen to be the induced union of complete graphs as it can be seen in Lemma 1.3.3 and Example 1.3.4.

Corollary 2.2.4 (Union of independent cliques)

Let $x \in S_G$ be a minimal solution of ϕ_G, and $H = H_x$ the depending subgraph. Let $b_{ij} \geq 2\min\{a_i, a_j\}$ for $ij \in E(G)$, and suppose H is the induced union of complete graphs.

Then H can be chosen to be an edgeless graph, and it yields

$$f(G)=\frac{1}{\max\left(\sum_{i\in S}\frac{1}{a_i}\right)}, \tag{2.13}$$

where the maximum is taken over all non-empty independent sets S of G. The minimum in $f(G)$ is attained by setting $x_i = \frac{f(G)}{a_i}$ for i in the optimal independent set S, and $x_i = 0$ otherwise.

PROOF: H may consist of multiple components named H_s for $s=1,\dots,r$ for a suitable $r\in\mathbb{N}$. Using Lemma 2.2.3 it follows $\frac{1}{f(G)} = \sum_s \frac{1}{f(H_s)}$.

Consider a component H_s on more than one vertex, and calculate $f(H_s)$ by Proposition 2.2.2. Because $b_{kl} \geq \min\{2a_k, 2a_l\} \geq 2\min_{i\in V(H_s)} a_i$ for all $kl \in E(H_s)$, the minimum can be attained by setting $x_i = 1$, and H_s can be chosen to have only one vertex.

Hence, H can be chosen to be an edgeless graph, and Proposition 2.2.1 completes the proof. ∎

It has not yet been possible for us to calculate $f(G)$ among more general preconditions than the treated ones. Merging the parameters b_{ij} for $ij \in E(G)$ to a single parameter b will help to attack this problem, but we would lose the benefit of the non-uniform parameters b's.

Therefore the meaning of the value $f(G)$ for the optimization problem remains obscure.

For concluding this section we proof Theorem 1:

Theorem 1

Let $w_i > 0$ for $i \in V(G)$. Then there is an independent set I of G such that

$$w(I) \geq \frac{\left(\sum\limits_{i \in V(G)} x_i\right)^2}{\sum\limits_{i \in V(G)} \frac{x_i^2}{w_i} + \sum\limits_{ij \in E(G)} \max\left\{\frac{1}{2w_i} + \frac{1}{w_j}, \frac{1}{w_i} + \frac{1}{2w_j}, \min\left\{\frac{2}{w_i}, \frac{2}{w_j}\right\}\right\} x_i x_j}$$

for all real $x_i \geq 0$ with $i \in V(G)$, satisfying $\sum_i x_i \neq 0$.

Proof: (of Theorem 1) Denote $b_{ij} = \max\left\{\frac{1}{2w_i} + \frac{1}{w_j}, \frac{1}{w_i} + \frac{1}{2w_j}, \min\left\{\frac{2}{w_i}, \frac{2}{w_j}\right\}\right\}$ for all $ij \in E(G)$.

In Remark 1.3.5 it is shown that for $b_{ij} \geq \max\left\{\frac{1}{2}a_i + a_j, a_i + \frac{1}{2}a_j\right\}$ the condition $C(P_3)$ is fulfilled. Thus, H can be chosen to be P_3-free. By Lemma 1.3.3, H is the induced union of complete graphs.

Because $b_{ij} \geq \min\{2a_i, 2a_j\}$ we can apply Corollary 2.2.4 and

$$f(G) = \frac{1}{\max\left(\sum\limits_{i \in S} \frac{1}{a_i}\right)}, \tag{2.14}$$

where the maximum is taken over all non-empty independent sets S of G.

It follows for all $z \in \mathbb{R}^n_{\geq 0}$ satisfying $z \neq 0$:

$$w(I) = \sum_{i \in I} \frac{1}{a_i} = \frac{1}{f(G)} \geq \frac{1}{\phi(y)} = \frac{\left(\sum\limits_{i \in V(G)} x_i\right)^2}{\sum\limits_{i \in V(G)} \frac{x_i^2}{w_i} + \sum\limits_{ij \in E(G)} b_{ij} x_i x_j}$$

where y is the normalised vector z, i. e. $y := \frac{z}{\sum_i z_i} \in S_G$. ∎

3 Remarks and Conclusions

Motivated by the quadratic problem formulation of the maximum clique problem of T. S. Motzkin and E. G. Straus [13], I investigated a generalised quadratic form on parameters a's and b's with regard to the minimum $f(G)$. Using a necessary condition for optimality, I was able to find a relation between the minimal solution $x \in S_G$ and its depending induced subgraph $H_x = G[\mathcal{X}_x]$ of G. With that result it was possible to describe the condition $C(F)$ that demand H to be $\mathcal{F}$-free. In some special cases the calculation of $f(G)$ was possible; for example the results of quadratic forms mentioned by T. S. Motzkin and E. G. Straus [13] could be generalised.

Using these results of quadratic forms, some applications on weighted graphs and on $\mathcal{F}$-free induced subgraphs of maximum weight could be received. The proceeding investigations and the conditions $C(F)$ were required to give a general framework, which is Theorem 2, for deriving an abundance of lower bounds for these induced subgraphs. Hereby, the known lower bound for the maximum weight independent set problem in Statement 2 by Gibbons et al. in [6] followed directly. The impact of this bound was presented by some investigations of other well known bounds. Finally, I was able to improve the result of Gibbons et al. and got Theorem 1.

It has not yet been possible for me to calculate $f(G)$ in most cases. Merging the parameters b_{ij} for $ij \in E(G)$ to a single parameter b will help to attack this problem, but one would lose the benefit of the non-uniform parameters b's. Lemma 2.2.3 for the components of H can be useful to solve the problem of calculating $f(G)$ for some more cases e. g. for G being a linear forest. The calculation of $f(G)$ for a simple path is a problem that I have not been able to solve yet. Therefore, the meaning of the value $f(G)$ for the optimization problem remains obscure. That would be an issue for future investigations.

Another related issue that seems worth investigating is the algebraic approach. Considering the necessary condition for optimality in Lemma 2.1.2 leads to the idea that it is equivalent to the homogeneous linear equation system for $V(H) = \{i_1, i_2, \dots, i_k\}$ and $f := f(G)$

$$\begin{pmatrix} 2a_{i_1} & \beta_{i_1 i_2} & \dots & \beta_{i_1 i_k} & -2 \\ \beta_{i_1 i_2} & 2a_{i_2} & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ \beta_{i_1 i_k} & \dots & \dots & 2a_{i_k} & -2 \end{pmatrix} \begin{pmatrix} x_{i_1} \\ \vdots \\ \vdots \\ x_{i_k} \\ f \end{pmatrix} = \overrightarrow{0},$$

where $\beta_{i_s i_t} = \begin{cases} \frac{1}{2} b_{i_s i_t}, & \text{if } i_s i_t \in E(H), \\ 0, & \text{else.} \end{cases}$

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